

# CHROMATIC POLYNOMIALS OF SIMPLICIAL COMPLEXES

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**ABSTRACT.** In this note we consider  $s$ -chromatic polynomials of finite simplicial complexes. The  $s$ -chromatic polynomials of simplicial complexes are higher dimensional analogues of chromatic polynomials for graphs.

## 1. INTRODUCTION

Let  $K$  be a finite simplicial complex with vertex set  $V(K) \neq \emptyset$  and let  $r \geq 1$  and  $s \geq 1$  be two natural numbers. A map  $\text{col}: V(K) \rightarrow \{1, 2, \dots, r\}$  is an  $(r, s)$ -coloring of  $K$  if there are no monochrome  $s$ -simplices in  $K$  [5]. We write  $\chi^s(K, r)$  for the number of  $(r, s)$ -colorings of  $K$ .

**Definition 1.1.** The  $s$ -chromatic polynomial of  $K$  is the function  $\chi^s(K, r)$  of  $r$ . The  $s$ -chromatic number of  $K$ ,  $\text{chr}^s(K)$ , is the minimal  $r \geq 1$  with  $\chi^s(K, r) > 0$ .

The theorem below shows that  $\chi^s(K, r)$  is indeed polynomial in  $r$  for fixed  $K$  and  $s$ . (By notational convention,  $[r]_i = r(r-1) \cdots (r-i+1)$  is the  $i$ th falling factorial in  $r$ .)

**Theorem 1.2.** The  $s$ -chromatic polynomial of  $K$  is

$$\chi^s(K, r) = \sum_{i=\text{chr}^s(K)}^{|V(K)|} S(K, i, s) [r]_i$$

where  $S(K, i, s)$  is the number of partitions of  $V(K)$  into  $i$  blocks containing no  $s$ -simplex of  $K$ .

For  $s = 1$ , an  $(r, 1)$ -coloring of  $K$  is a usual graph coloring,  $\chi^1(K, r)$  is the usual chromatic polynomial, and  $\text{chr}^1(K)$  the usual chromatic number of the 1-skeleton of  $K$ . In general,  $\chi^s(K, r)$  depends only on the  $s$ -skeleton of  $K$ . Although the higher  $s$ -chromatic polynomials for simplicial complexes are analogues of 1-chromatic polynomials for graphs we shall shortly see that there are structural differences between the cases  $s =$  and  $s > 1$ .

Figure 1 shows a triangulation MB of the Möbius band. To the left is a  $(5, 1)$ - and to the right a  $(2, 2)$ -coloring of MB. The chromatic polynomials and chromatic numbers<sup>1</sup> of MB are

$$\chi^s(\text{MB}, r) = \begin{cases} r^5 - 10r^4 + 35r^3 - 50r^2 + 24r & s = 1 \\ r^5 - 5r^3 + 5r^2 - r & s = 2 \\ r^5 & s \geq 3 \end{cases} \quad \text{chr}^s(\text{MB}) = \begin{cases} 5 & s = 1 \\ 2 & s = 2 \\ 1 & s \geq 3 \end{cases}$$

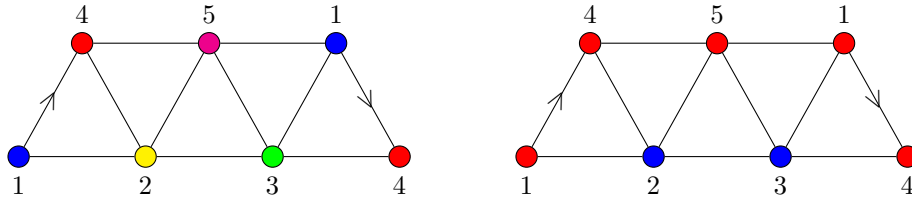


FIGURE 1. A  $(5, 1)$ -coloring and a  $(2, 2)$ -coloring of a 5-vertex triangulated Möbius band MB

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<sup>1</sup>The computations behind the examples of this note were carried out in the computer algebra system Magma [3].

1.1. **Notation.** We shall use the following notation throughout the paper:

$K$ : a finite simplicial complex

$K^s$ : the  $s$ -skeleton of  $K$

$F^s(K)$ : the set of  $s$ -simplices  $K$

$\#V$  or  $|V|$ : the number of elements in the finite set  $V$

$V(K)$ : the vertex set  $\bigcup K$  of  $K$  and  $m(K) = |V(K)|$  is the number of vertices in  $K$

$D[V]$ : the complete simplicial complex of *all* subsets of the finite set  $V$

$[m]$ : the finite set  $\{1, \dots, m\}$  of cardinality  $m$

$[r]_i$ : the  $i$ th falling factorial polynomial  $[r]_i = i! \binom{r}{i}$  in  $r$

$P(a, b)$ : the open interval  $(a, b)$  in the poset  $P$

## 2. THREE WAYS TO THE $s$ -CHROMATIC POLYNOMIAL OF A SIMPLICIAL COMPLEX

In this section we present three different approaches to the  $s$ -chromatic polynomial  $\chi^s(K, r)$ :

- Theorem 2.5 via 1-chromatic polynomials of graphs;
- Theorem 2.25 via the Möbius function for the  $s$ -chromatic lattice;
- Theorem 1.2 via the simplicial  $s$ -Stirling numbers of the second kind.

2.1. **Block-connected  $s$ -independent vertex partitions.** Let  $s \geq 1$  be a natural number.

**Definition 2.1.** Let  $B \subset V(K)$  be a set of vertices of  $K$ . Then

- $B$  is  $s$ -independent if  $B$  contains no  $s$ -simplex of  $K$ ;
- $B$  is connected if  $K \cap D[B]$  is a connected simplicial complex;
- the connected components of  $B$  are the maximal connected subsets of  $B$ .

**Definition 2.2.** Let  $P$  be a partition of  $V(K)$ .

- The graph  $G_0(P)$  of  $P$  is the simple graph whose vertices are the blocks of  $P$  and with two blocks connected by an edge if their union is connected;
- The block-connected refinement  $P_0$  of  $P$  is the refinement whose blocks are the connected components of the blocks of  $P$ ;
- $P$  is block-connected if the blocks of  $P$  are connected (ie if  $P = P_0$ ).

**Lemma 2.3.** Let  $P$  be a partition of  $V(K)$ . If two different blocks of the block-connected refinement  $P_0$  are connected by an edge in the graph  $G_0(P_0)$  of  $P$  then they lie in different blocks of  $P$ .

*Proof.* The connected components of the blocks of  $P$  are maximal with respect to connectedness.  $\square$

**Definition 2.4.**  $\text{BCP}^s(K)$  is the set of all block-connected  $s$ -independent partitions of  $V(K)$ .

Recall that  $\chi^1(G_0(P), r)$  is the 1-chromatic polynomial of the simple graph  $G_0(P)$  of the partition  $P$ .

**Theorem 2.5.** The  $s$ -chromatic polynomial for  $K$  is the sum

$$\chi^s(K, r) = \sum_{P \in \text{BCP}^s(K)} \chi^1(G_0(P), r)$$

of the 1-chromatic polynomials and the  $s$ -chromatic number of  $K$  is the minimum

$$\text{chr}^s(K) = \min_{P \in \text{BCP}^s(K)} \text{chr}^1(G_0(P))$$

of the 1-chromatic numbers for the graphs of all the block-connected  $s$ -independent partitions of  $V(K)$ .

*Proof.* Let  $\text{col}: V(K) \rightarrow [r]$  be an  $(r, s)$ -coloring of  $K$ . The monochrome partition  $P(\text{col})$  of  $V(K)$  is the  $s$ -independent partition whose blocks are the nonempty monochrome sets of vertices  $\{\text{col} = i\}$  for  $i \in [r]$ . The block-connected refinement  $P(\text{col})_0$  of the monochrome partition is a block-connected  $s$ -independent partition of  $K$ . The original coloring  $\text{col}$  of  $K$  is also a coloring of the graph  $G_0(P(\text{col})_0)$  of  $P(\text{col})_0$  for, by Lemma 2.3, distinct vertices of 1-simplices of this graph have distinct colors. We have shown that any  $(r, s)$ -coloring  $\text{col}$  of  $K$  induces an  $(r, 1)$ -coloring  $\text{col}_0$  of the graph  $G_0(P(\text{col})_0)$  of the block-connected refinement of the monochrome partition.

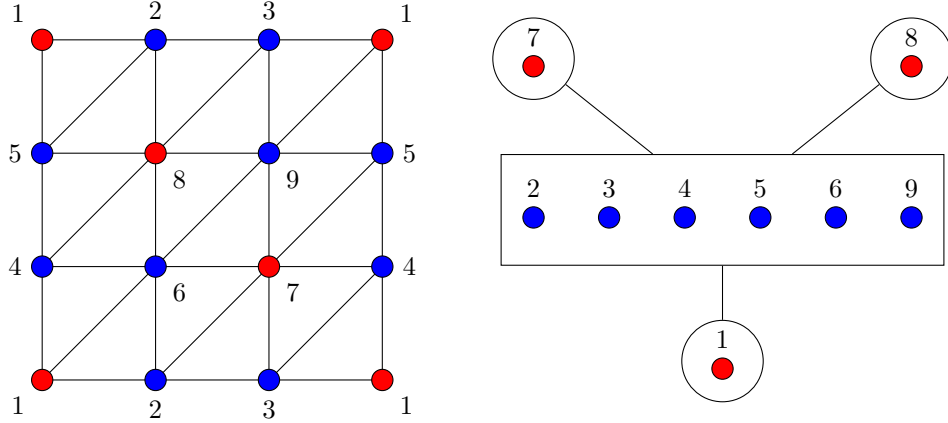
Let  $P \in \text{BCP}^s(K)$  be a block-connected  $s$ -independent partition of  $V(K)$  and  $\text{col}_0: P \rightarrow \{1, \dots, r\}$  an  $(r, 1)$ -coloring of its graph  $G_0(P)$ . Then  $\text{col}_0$  determines a map  $\text{col}: V(K) \rightarrow [r]$  that is constant on the blocks of  $P$ . An  $s$ -simplex of  $K$  can not be monochrome under  $\text{col}$  as it intersects at least two different blocks of  $P$  connected by an edge of  $G_0(P)$ . Thus  $\text{col}$  is an  $(r, s)$ -coloring of  $K$ .

These two constructions are inverses of each other.  $\square$

**Remark 2.6** (The minimal block-connected  $s$ -independent partition). Let  $C_0 = \{\{v\} \mid v \in V(K)\}$  be the block-connected  $s$ -independent partition of  $V(K)$  whose blocks are singletons. The graph  $G_0(C_0) = K^1$  is the 1-skeleton of  $K$ . Thus the 1-chromatic polynomial of the 1-skeleton of  $K$  is always one of the polynomials in the sum of Theorem 2.5. If  $K$  is 1-dimensional,  $\text{BCP}^1(K)$  consists only of the partition  $C_0$  and Theorem 2.5 simply says that the 1-chromatic polynomial of a simplicial complex is the 1-chromatic polynomial of its 1-skeleton.

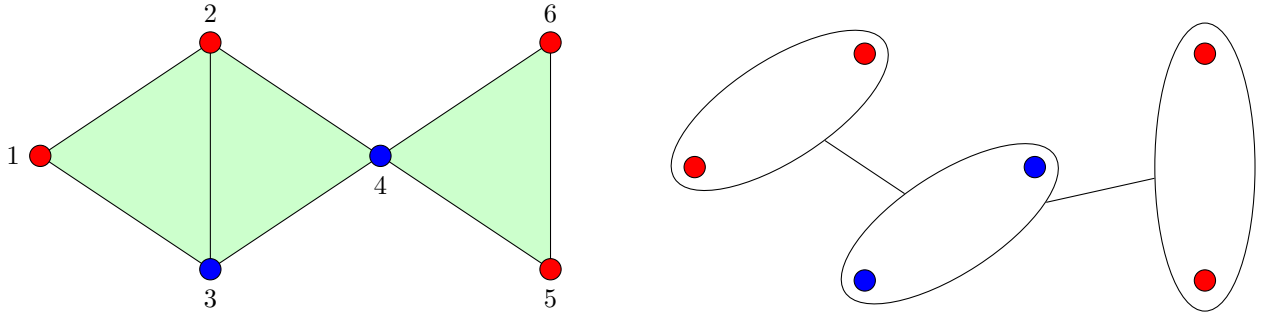
**Example 2.7** (The block-connected 2-independent partitions for  $D[3]$ ). The 2-simplex  $D[3]$  has 4 block-connected 2-independent partitions  $C_0, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\},$  and  $\{\{3\}, \{1, 2\}\}$ . The graph of  $C_0$  is the complete graph  $K_3$ , the 1-skeleton of  $D[3]$ . The graphs of the other three partitions are all the complete graph  $K_2$ . Thus the 2-chromatic polynomial of  $D[3]$  is  $\chi^2(D[3], r) = \chi^1(K_3, r) + 3\chi^1(K_2, r) = [r]_3 + 3[r]_2 = [r]_2(r+1) = r^3 - r$  and the 2-chromatic number is  $\text{chr}^2(D[3]) = 2$ .

**Example 2.8** (A  $(2, 2)$ -coloring and the graph of the block-connected refinement of its monochrome partition). The picture below illustrates a  $(2, 2)$ -coloring of a 9-vertex triangulation of the torus



and its corresponding graph. There are 6937 block-connected partitions of the vertex set, and 3 of them has the graph shown above. The 2-chromatic polynomial is  $21[r]_2 + 742[r]_3 + 3747[r]_4 + 4908[r]_5 + 2295[r]_6 + 444[r]_7 + 36[r]_8 + [r]_9 = [r]_2(r^7 + r^6 - 17r^5 + 10r^4 + 82r^3 - 116r^2 - 23r + 67)$  and the 2-chromatic number is 2.

**Example 2.9** (The  $(r, 2)$ -colorings of a simplicial complex  $K$ ). Let  $K$  be the pure 2-dimensional complex with facets  $F^2(K) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{4, 5, 6\}\}$ .



The picture shows a  $(2, 2)$ -coloring of  $K$  and the corresponding  $(2, 1)$ -coloring of the associated graph,  $G_0(P_0)$ , the block connected refinement of the monochrome partition  $P = \{\{1, 2, 5, 6\}, \{3, 4\}\}$ . Table 1 shows the graphs  $G_0(P)$  for all block connected partitions  $P \in \text{BCP}^2(K)$ . For each graph, the table records its 1-chromatic polynomial and its 1-chromatic number. The 2-chromatic polynomial of  $K$  is  $\chi^2(K, 2) = 15[r]_2 + 73[r]_3 + 62[r]_4 + 15[r]_5 + [r]_6 = [r]_2(r-1)(r+1)(r^2+r-1)$  and the 2-chromatic number is  $\text{chr}^2(K) = 2$ .

**Example 2.10** (The  $(r, 2)$ -colorings of the Möbius band). The set  $\text{BCP}^2(\text{MB})$  of block-connected 2-independent partitions of the triangulated Möbius band MB (Figure 1) has 36 elements. There are 5, 5, 15, 10, 1 partitions in  $\text{BCP}^2(\text{MB})$  realizing the partitions  $[3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1]$  of the integer  $|V(\text{MB})| = 5$ . All associated graphs are complete graphs. This yields the 2-chromatic polynomial  $\chi^2(\text{MB}, r) = 5[r]_2 + 20[r]_3 + 10[r]_4 + [r]_5 = [r]_2(r^3 + r^2 - 4r + 1) = r^5 - 5r^3 + 5r^2 - r$  and the 2-chromatic number is  $\text{chr}^2(\text{MB}) = 2$ .

**Remark 2.11** (The  $\mathcal{S}$ -chromatic polynomial of  $K$ ). Let  $\mathcal{S}$  be a set of connected subcomplexes of  $K$ . A set  $B \subset V(K)$  of vertices is  $\mathcal{S}$ -independent if  $B$  is not a superset of any member of  $\mathcal{S}$ . Let  $\text{BCP}^{\mathcal{S}}(K)$  be the set of

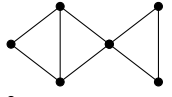
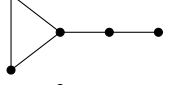
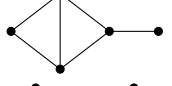

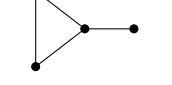
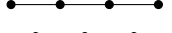

# in $\text{BCP}^2(K)$	$G_0(P)$	$\chi^1(G_0(P), r)$	$\text{chr}^1(G_0(P))$
1		$r(r-1)^2(r-2)^3$	3
1		$r(r-1)^3(r-2)$	3
3		$r(r-1)^2(r-2)^2$	3
4		$r(r-1)^2(r-2)^2$	3
16		$r(r-1)^2(r-2)$	3
3		$r(r-1)^3$	2
12		$r(r-1)^2$	2

TABLE 1. The graphs for the block-connected partitions in  $\text{BCP}^2(K)$ 

$\mathcal{S}$ -independent partitions of  $V(K)$ . An  $(r, \mathcal{S})$ -coloring is a map  $V(K) \rightarrow \{1, \dots, r\}$  such that  $\#\text{col}(S) > 1$  for all  $S \in \mathcal{S}$ . The number of  $(r, \mathcal{S})$ -colorings of  $K$  is

$$\chi^{\mathcal{S}}(K, r) = \sum_{P \in \text{BCP}^{\mathcal{S}}(K)} \chi^1(G_0(P), r)$$

as one sees by an obvious generalization of Theorem 2.5. An  $(r, s)$ -coloring of  $K$  is an  $(r, \mathcal{S})$ -coloring of  $K$  where  $\mathcal{S} = F^s(K)$  is the set of  $s$ -simplices.

**2.2. The  $s$ -chromatic linear program.** Read [9, §10] explains how to construct a linear program with minimal value equal to the  $s$ -chromatic number  $\text{chr}^s(K)$  of  $K$ .

**Definition 2.12.**  $M^s(K)$  is the set of all maximal  $s$ -independent subsets of  $V(K)$ .

Let  $A$  be the  $(m(K) \times |M^s(K)|)$ -matrix

$$A(v, M) = \begin{cases} 1 & v \in M \\ 0 & v \notin M \end{cases}$$

recording which vertices  $v \in V(K)$  belong to which maximal  $s$ -independent sets  $M \in M^s(K)$ . Now the  $s$ -chromatic number

$$\text{chr}^s(K) = \min \left\{ \sum_{M \in M^s(K)} x(M) \mid x: M^s(K) \rightarrow \{0, 1\}, \forall v \in V(K): \sum_{M \in M^s(K)} A(v, M)x(M) \geq 1 \right\}$$

is the minimal value of the objective function  $\sum_{M \in M^s(K)} x(M)$  in  $|M^s(K)|$  variables  $x: M^s(K) \rightarrow \{0, 1\}$ , taking values 0 or 1, and  $m(K)$  constraints  $\sum_{M \in M^s(K)} A(v, M)x(M) \geq 1, v \in V(K)$ .

**2.3. The  $s$ -chromatic lattice.** Our approach here simply follows Rota's classical method for computing chromatic polynomials from Möbius functions of lattices [10, §9]. We need some terminology in order to characterize the monochrome loci for colorings of  $K$ . Recall that  $F^s(K)$  is the set of  $s$ -simplices of  $K$ .

**Definition 2.13.** Let  $S \subset F^s(K)$  be a set of  $s$ -simplices of  $K$ .

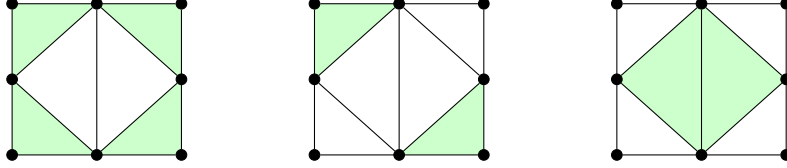
- The equivalence relation  $\sim$  is the smallest equivalence relation in  $S$  such that  $s_1 \cap s_2 \neq \emptyset \implies s_1 \sim s_2$  for all  $s_1, s_2 \in S$ ;
- the connected components of  $S$  are the equivalence classes under  $\sim$ ;
- $\pi_0(S)$  is the set of connected components of  $S$ ;
- $S$  is connected if it has at most one component;
- $V(S) = \bigcup S$  is the vertex set of  $S$ ;
- $\pi(S)$  is the partition of  $V(K)$  whose blocks are the vertex sets of the connected components of  $S$  together with the singleton blocks  $\{v\}$ ,  $v \in V(K) - V(S)$ , of vertices not in any simplex in  $S$ ;

- $S$  is closed if  $S$  contains any  $s$ -simplex in  $K$  contained in the vertex set of  $S$ , ie if
$$\{\sigma \in F^s(K) \mid \sigma \subset V(S)\} = S$$
- the closure of  $S$  is the smallest closed set of  $s$ -simplices containing  $S$ .

For instance, the empty set  $S = \emptyset$  of 0  $s$ -simplices is connected with 0 connected components. If  $K = D[4]$ , the set  $\{\{1, 2\}, \{2, 4\}\}$  of 1-simplices is connected while  $\{\{1, 2\}, \{3, 4\}\}$  has the two components  $\{\{1, 2\}\}$  and  $\{\{3, 4\}\}$ .

A set of  $s$ -simplices is closed if and only if it equals its closure. For instance in  $F^2(D[5])$ , the set  $\{\{1, 2, 3\}, \{3, 4, 5\}\}$  is not closed because its closure is the set of all 2-simplices in  $D[5]$ . The empty set of  $s$ -simplices, any set of just one  $s$ -simplex, and any set of disjoint  $s$ -simplices are closed.

In this picture the green set of 2-simplices is



connected and not closed, closed and not connected, closed and connected, respectively.

The partition  $\pi(S)$  has  $|\pi(S)| = |\pi_0(S)| + m(K) - |V(S)|$  blocks.

**Lemma 2.14.** *Let  $S$  be a set of  $s$ -simplices in  $K$  and  $S_0$  a connected component of  $S$ . Then  $S_0$  is closed if and only if*

$$\{\sigma \in F^s(K) \mid \sigma \subset V(S_0)\} \subset S_0$$

*Proof.* Since the condition is certainly necessary we only need to see that it is sufficient. Let  $\sigma$  be an  $s$ -simplex in  $K$  with all its vertices in  $V(S_0)$ . Then  $\sigma$  lies in  $S$  by assumption. But  $\sigma$  is equivalent to all elements of the equivalence class  $S_0$ . Thus  $\sigma \in S_0$ .  $\square$

**Lemma 2.15.** *Let  $S$  and  $T$  be sets of  $s$ -simplices in  $K$ .*

- (1) *If  $S$  and  $T$  are closed, so is  $S \cap T$ .*
- (2) *If  $S$  and  $T$  have closed connected components, so does  $S \cap T$*

*Proof.* (1) Let  $\sigma$  be an  $s$ -simplex of  $K$  and suppose that  $\sigma \subset V(S \cap T)$ . Then  $\sigma \subset V(S)$  and  $\sigma \subset V(T)$  so that  $\sigma \in S$  and  $\sigma \in T$  as  $S$  and  $T$  are closed.

(2) Let  $R$  be a connected component of  $S \cap T$ . Let  $S_0$  be the connected component of  $S$  containing  $R$  and  $T_0$  be the connected component of  $T$  containing  $R$ . Then  $R \subset S_0 \cap T_0$ . Suppose that  $\sigma \in F^s(K)$  is an  $s$ -simplex with  $\sigma \subset V(R)$ . Then  $\sigma \subset V(S_0 \cap T_0)$  so  $\sigma \in S_0 \cap T_0$  by (1) as the connected components  $S_0$  and  $T_0$  are assumed to be closed. In particular,  $\sigma \in S \cap T$ . According to Lemma 2.14, the connected component  $R$  is closed.  $\square$

**Definition 2.16.** *The  $s$ -chromatic lattice of  $K$  is the set  $L^s(K)$  of all subsets of  $F^s(K)$  with closed connected components.  $L^s(K)$  is a partially ordered by set inclusion.*

The set  $L^s(K)$  contains the empty set  $\emptyset$  of  $s$ -simplices and the set  $F^s(K)$  of all  $s$ -simplices. These two elements of  $L^s(K)$  are distinct when  $K$  has dimension at least  $s$ .

**Corollary 2.17.**  *$L^s(K)$  is a finite lattice with  $\hat{0} = \emptyset$ ,  $\hat{1} = F^s(K)$ , and meet  $S \wedge T = S \cap T$ .*

*Proof.* If  $S, T \in L^s(K)$  then  $S \cap T$  is also in  $L^s(K)$  by Lemma 2.15 and this is clearly the greatest lower bound of  $S$  and  $T$ . It is now a standard result that  $L^s(K)$  is a finite lattice [12, Proposition 3.3.1]. The join  $S \vee T$  of  $S, T \in L^s(K)$  is the intersection of all supersets  $U \in L^s(K)$  of  $S \cup T$ .  $\square$

**Example 2.18** (The  $s$ -chromatic lattice  $L^s(D[m])$ ). The closed and connected elements of the  $s$ -chromatic lattice  $L^s(D[m])$  of the complete simplex  $D[m]$  on  $m > s$  vertices are  $\emptyset$  and the  $\binom{m}{k}$  sets  $F^s(D[k])$  of all  $s$ -simplices in the subcomplexes  $D[k]$  for  $s < k \leq m$ . The map  $S \rightarrow \pi(S)$  is an isomorphism between the lattice  $L^s(D[m])$  and the lattice, ordered by refinement, of all partitions of the set  $[m]$  into blocks of size  $> s$  or 1. The least element,  $\hat{0} = (1) \cdots (m)$ , is the partition with  $m$  blocks and the greatest element,  $\hat{1} = (1 \cdots m)$ , the partition with 1 block.  $L^s(D[m])$  is not a graded lattice [12, p 99] in general when  $s \geq 2$ . To see this, observe that the 2-chromatic lattices  $L^2(D[3])$ ,  $L^2(D[4])$ , and  $L^2(D[4])$  are graded but the lattice  $L^2(D[6])$  is not graded as it contains two maximal chains

$$\begin{aligned} \hat{0} &= (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (1234)(5)(6) < (12345)(6) < (123456) = \hat{1} \\ \hat{0} &= (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (123)(456) < (123456) = \hat{1} \end{aligned}$$

of unequal length. In contrast, the 1-chromatic lattice of any finite simplicial complex is always graded and even geometric [10, §9, Lemma 1].

**Remark 2.19** (The Möbius function for the  $s$ -chromatic lattices  $L^s(D[m])$ ). Our discussion of the Möbius function for the lattice  $L^s(D[m])$  echoes the exposition of the Möbius function for the geometric lattice  $L^1(D[m])$  of all partitions from [12, Example 3.10.4].

Let  $w: [m] \rightarrow \mathbf{N}$  be a function that to every element of  $[m]$  associates a natural number, thought of as a weight function. We write  $w = 1^{i_1} 2^{i_2} \dots r^{i_r}$ , or something similar, for the weight function  $w$  defined on the set  $[m]$  of cardinality  $m = \sum_j i_j$  and mapping  $i_j$  elements to  $j$  for  $1 \leq j \leq r$ . The map  $w$  extends to a map, also called  $w$ , defined on the set of all nonempty subsets  $X$  of  $[m]$  given by  $w(X) = \sum_{x \in X} w(x)$ . Let  $L_m^s(w)$  be the lattice of all partitions of the set  $[m]$  into blocks  $X$  that are singletons or have weight  $w(X) > s$ . The non-singleton blocks of the meet  $\sigma \wedge \tau$  of two partitions  $\sigma, \tau \in L_m^s(w)$  are the subsets of weight  $> s$  of the form  $S \cap T$  where  $S$  is a block in  $\sigma$  and  $T$  a block in  $\tau$ . Write  $\mu_m^s(w)$  for the Möbius function of  $L_m^s(w)$ .

In particular,  $L_m^s(1^m)$  is a synonym for  $L^s(D[m])$  and we are primarily interested in the Möbius function  $\mu_m^s(1^m)$  of the uniform weight  $w = 1^m$ . However, the computation of this Möbius function will involve the Möbius functions of other weights as well. We shall therefore discuss the Möbius functions  $\mu_m^s(w)$  for general weight functions  $w$ .

Suppose that  $\sigma \in L_m^s(w)$ ,  $\sigma < \hat{1}$ , is a partition of  $[m]$  into singleton blocks or blocks of weight  $> s$ . Let  $w(\sigma)$  be the restriction of  $w$  to the set of blocks of  $\sigma$ . Thus  $w(\sigma)(X) = \sum_{x \in X} w(x)$  for any block  $X$  of  $\sigma$ . Then the interval

$$L_m^s(w) \supset [\sigma, \hat{1}] = L_{|\sigma|}^s(w(\sigma))$$

so that  $\mu_m^s(w)(\sigma, \hat{1}) = \mu_{|\sigma|}^s(w(\sigma))(\hat{0}, \hat{1})$ . More generally, suppose that  $\sigma < \tau$  for some  $\tau \in L_m^s(w)$ . Assume that the partition  $\tau$  has blocks  $\tau_j$ . Let  $\sigma_j$  be the set of those blocks of  $\sigma$  that intersect the block  $\tau_j$  of  $\tau$ . Let  $w(\sigma_j)$  be the restriction of  $w(\sigma)$  to  $\sigma_j$ . Then the interval

$$L_m^s(w) \supset [\sigma, \tau] = \prod_j L_{|\sigma_j|}^s(w(\sigma_j))$$

and therefore the value of the Möbius function on the pair  $(\sigma, \tau)$

$$\mu_m^s(w)(\sigma, \tau) = \prod_j \mu_{|\sigma_j|}^s(w(\sigma_j))(\hat{0}, \hat{1})$$

by the product theorem for Möbius functions [12, Proposition 3.8.2]. We conclude that the complete Möbius functions on all the lattices  $L_m^s(w)$ , are actually determined by the values  $\mu_m^s(w)(\hat{0}, \hat{1})$  of these Möbius functions on just  $(\hat{0}, \hat{1})$ . See Equation (2.36) for more information about these Euler characteristics.

For the following it is convenient to name the elements of the domain  $[m]$  of  $w$  so that the element  $m$  carries minimal weight. Assume that  $a_m = (1 \dots m-1)(m)$  is an element of  $L_m^s(w)$ , ie that  $w(1) + \dots + w(m-1) > s$ . We shall determine the set of lattice elements  $x$  with  $x \wedge a_m = \hat{0}$ . There is only one solution to this equation with  $x \leq a_m$  and that is  $x = \hat{0}$ . As the other solutions satisfy  $x \not\leq a_m$ , they must have a block that contains  $m$  and at least one other element. It follows that the solutions  $x \neq \hat{0}$  are all elements of the form

$$x = (x_1 \dots x_t m)(\cdot) \dots (\cdot) \text{ with } \begin{cases} w(x_1) > s - w(m) & t = 1 \\ s \geq w(x_1) + \dots + w(x_t) > s - w(m) & t > 1 \end{cases}$$

where all blocks but the unique block containing  $m$  are singletons. There are  $t+1$  elements in the block containing  $m$  where  $t$  is some number in the range  $1 \leq t \leq s$ . (All the solutions  $x \neq \hat{0}$  are atoms in the lattice  $L_m^s(w)$ .) Since we are in a lattice, the Möbius function satisfies the equation [12, Corollary 3.9.3]

$$\mu_m^s(w)(\hat{0}, \hat{1}) = - \sum_{\substack{x \wedge a_m = \hat{0} \\ x \neq \hat{0}}} \mu_m^s(w)(x, \hat{1})$$

which translates to

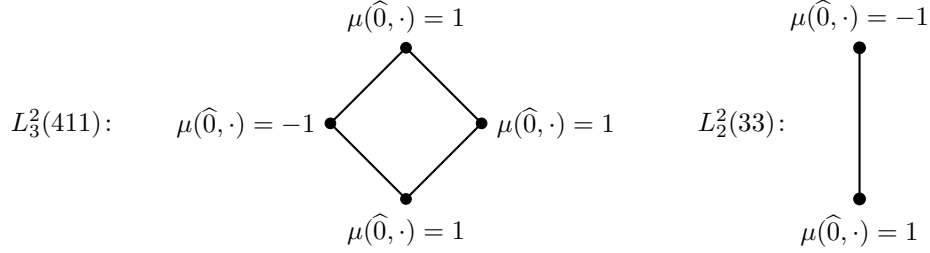
$$\begin{aligned} (2.20) \quad \mu_m^s(w)(\hat{0}, \hat{1}) &= - \sum_{\substack{x \wedge a_m = \hat{0} \\ x \neq \hat{0}}} \mu_{|x|}^s(w(x))(\hat{0}, \hat{1}) = \\ &- \sum_{\substack{1 \leq x_1 \leq m-1 \\ w(x_1) > s - w(m)}} \mu_{m-1}^s(w(x_1 m)w(\cdot) \dots w(\cdot))(\hat{0}, \hat{1}) - \sum_{1 < t \leq s} \sum_{\substack{1 \leq x_1, \dots, x_t \leq m-1 \\ s \geq w(x_1) + \dots + w(x_t) > s - w(m)}} \mu_{m-t}^s(w(x_1 \dots x_t m)w(\cdot) \dots w(\cdot))(\hat{0}, \hat{1}) \end{aligned}$$

This describes a recursive procedure for computing all values of the Möbius function on the weight lattices  $L_m^s(w)$ .

As an illustration we compute  $\mu_6^2(1^6)(\widehat{0}, \widehat{1})$ . Using (2.20) twice gives

$$\mu_6^2(1^6)(\widehat{0}, \widehat{1}) = -10\mu_4^2(3111)(\widehat{0}, \widehat{1}) = 10(\mu_3^2(411)(\widehat{0}, \widehat{1}) + \mu_2^2(33)(\widehat{0}, \widehat{1}))$$

The lattices  $L_4^2(411)$  and  $L_2^2(33)$  have 4 and 2 elements, respectively, and they look like



so that  $\mu_3^2(411)(\widehat{0}, \widehat{1}) = 1$  and  $\mu_2^2(33)(\widehat{0}, \widehat{1}) = -1$ . Therefore  $\mu_6^2(1^6)(\widehat{0}, \widehat{1}) = 0$ .

We remind the reader of the well-known fact that  $\mu_m^s(w)(\widehat{0}, \widehat{1})$  is the reduced Euler characteristic of the open interval  $L_m^s(w)(\widehat{0}, \widehat{1})$  between  $\widehat{0}$  and  $\widehat{1}$  in the lattice  $L_m^s(w)$ .

**Proposition 2.21.** [10, §6] [12, Proposition 3.8.5] *Let  $x < y$  be two elements in a finite poset. The value of the Möbius function on the pair  $(x, y)$  is the reduced Euler characteristic of the open interval  $(x, y)$ .*

*Proof.* Write  $\mu$  be the Möbius function of  $P$  and  $E$  for Euler characteristic. The closed interval from  $x$  to  $y$  has Euler characteristic 1 since it has a smallest element. Thus

$$\begin{aligned} 1 = E([x, y]) &= \sum_{a, b \in [x, y]} \mu(a, b) = \sum_{a, b \in (x, y)} \mu(a, b) + \sum_{a \in [x, y]} \mu(a, y) + \sum_{b \in [x, y]} \mu(x, b) - \mu(x, y) \\ &= E((x, y)) + 0 + 0 - \mu(x, y) = E((x, y)) - \mu(x, y) \end{aligned}$$

or  $\mu(x, y) = \widetilde{E}((x, y))$ . □

For  $1 \leq s \leq m + 1$  let  $B(m, s)$  be the graded poset of nonempty subsets of  $[m]$  of cardinality less than  $s$ .

**Lemma 2.22.** *The reduced Euler characteristic of  $B(m, s)$  is*

$$\widetilde{E}(B(m, s)) = (-1)^s \binom{m-1}{s-1}, \quad 1 \leq s \leq m+1$$

*Proof.* It is rather easy to get the recurrence relation

$$E(B(m, 2)) = m$$

$$E(B(m, s)) = E(B(m, s-1)) + \binom{m}{s-1} \sum_{j=1}^{s-1} (-1)^{s-1-j} \binom{s-1}{j}, \quad 2 < s < 2+m$$

Since the sum of binomial coefficients has value  $(-1)^s$ , we get the recurrence relation

$$\widetilde{E}(B(m, 2)) = m - 1$$

$$\widetilde{E}(B(m, s)) = \widetilde{E}(B(m, s-1)) + (-1)^s \binom{m}{s-1}, \quad 2 < s < 2+m$$

for the reduced Euler characteristic. The claim of the lemma follows immediately. □

**Example 2.23** (Reduced Euler characteristics of the  $s$ -chromatic lattice intervals  $L_m^s(w)(\widehat{0}, \widehat{1})$ ). The reduced Euler characteristics  $\mu_m^s(1^m)(\widehat{0}, \widehat{1}) = \widetilde{E}(L_m^s(1^m)(\widehat{0}, \widehat{1}))$ ,  $m \geq s+2$ , for  $s = 1, 2, \dots, 8$  are

- 2, -6, 24, -120, 720, -5040, 40320, -362880, 3628800, -39916800, 479001600, -6227020800, 87178291200, ...
- 3, -6, 0, 90, -630, 2520, 0, -113400, 1247400, -7484400, 0, 681080400, -10216206000, 81729648000, ...
- 4, -10, 20, -70, 560, -4200, 25200, -138600, 924000, -8408400, 84084000, -798798000, 7399392000, ...
- 5, -15, 35, -70, 0, 2100, -23100, 173250, -1051050, 5255250, -15765750, -105105000, 2858856000, ...
- 6, -21, 56, -126, 252, -924, 11088, -126126, 1093092, -7693686, 46414368, -254438184, 1492322832, ...
- 7, -28, 84, -210, 462, -924, 0, 42042, -630630, 6390384, -51459408, 351639288, -2118412296, 11406835440, ...
- 8, -36, 120, -330, 792, -1716, 3432, -12870, 205920, -3150576, 35706528, -322583976, 2460949920, ...
- 9, -45, 165, -495, 1287, -3003, 6435, -12870, 0, 787644, -14965236, 191222460, -1920538620, ...



The first sequence,  $\mu_m^1(1^m)(\widehat{0}, \widehat{1})$ ,  $m \geq 2$ , is the sequence  $(-1)^{m-1}(m-1)!$  of reduced Euler characteristics of the lattice of partitions of  $[m]$  [12, Example 3.10.4]. The second sequence,  $\mu_m^2(1^m)(\widehat{0}, \widehat{1})$ ,  $m \geq 3$ , seems to coincide with first terms of the sequence A009014 from The On-Line Encyclopedia of Integer Sequences (OES). The remaining 6 sequences apparently do not match any sequences of the OES.

The first  $s$  terms of these sequences are signed binomial coefficients. This is because the interval  $(\widehat{0}, \widehat{1})$  in  $L^s(D[m])$  is isomorphic to the opposite of the poset  $B(m, m-s)$  when  $s+2 \leq m \leq 2s+1$ . Thus the reduced Euler characteristic

$$\mu_m^s(1^m)(\widehat{0}, \widehat{1}) = \widetilde{E}(B(m, m-s)) = (-1)^{m-s} \binom{m-1}{s}, \quad s+2 \leq m \leq 2s+1,$$

according to Lemma 2.22.

The first terms of the sequence  $\mu_m^2(3^1 1^{m-1})(\widehat{0}, \widehat{1})$ ,  $m \geq 3$ , of reduced Euler characteristics of the weighted lattice intervals  $L_m^2(3^1 1^{m-1})(\widehat{0}, \widehat{1})$ ,

$$1, 0, -6, 30, -90, 0, 2520, -22680, 113400, 0, -7484400, 97297200, -681080400, 0, 81729648000, -1389404016000, \dots$$

seem to coincide up to sign with first terms of the sequence A009775 from OES. The sequence of reduced Euler characteristics  $\mu_m^2(3^2 1^{m-2})(\widehat{0}, \widehat{1})$ ,  $m \geq 3$ , of the lattice interval  $L_m^2(3^2 1^{m-2})(\widehat{0}, \widehat{1})$ ,

$$2, -4, 6, 6, -120, 720, -2520, -2520, 136080, -1360800, 7484400, 7484400, \\ -778377600, 10897286400, -81729648000, -81729648000, 13894040160000, \dots$$

apparently does not match any sequence in the OES.

Define the  $s$ -monochrome set of a map  $\text{col}: V(K) \rightarrow [r] = \{1, \dots, r\}$  to be the set

$$M^s(\text{col}) = \{\sigma \in F^s(K) \mid |\text{col}(\sigma)| = 1\}$$

of all monochrome  $s$ -simplices in  $K$ . The map  $\text{col}$  is an  $(r, s)$ -coloring of  $K$  if and only if  $M^s(\text{col}) = \emptyset$ .

**Lemma 2.24.** *The  $s$ -monochrome set  $M^s(\text{col})$  of any map  $\text{col}: V(K) \rightarrow [r]$  is an element of the  $s$ -chromatic lattice  $L^s(K)$ .*

*Proof.* Let  $S$  be a connected component of  $M^s(\text{col})$ . Since  $S$  is connected, all vertices in  $S$  have the same color. Let  $\sigma \in F^s(K)$  be an  $s$ -simplex of  $K$  such that  $\sigma \subset V(S)$ . The  $\sigma$  is monochrome:  $\sigma \in M^s(\text{col})$ . By Lemma 2.14,  $S$  is closed.  $\square$

**Theorem 2.25.** *The number of  $(r, s)$ -colorings of  $K$  is*

$$\chi^s(K, r) = \sum_{T \in L^s(K)} \mu(\widehat{0}, T) r^{|\pi(T)|}$$

where  $\mu$  the Möbius function for the  $s$ -chromatic lattice  $L^s(K)$ .

*Proof.* For any  $B \in L^s(K)$ , let  $\chi(K, r, s, B)$  be the number of maps  $\text{col}: V(K) \rightarrow [r]$  with  $M^s(\text{col}) = B$ . We want to determine  $\chi(K, r, s, \emptyset) = \chi^s(K, r)$ . For any  $A \in L^s(K)$ ,

$$r^{|\pi(A)|} = \sum_{A \leq B} \chi(K, r, s, B)$$

because there are  $r^{|\pi_0(A)|} r^{m(K) - |V(A)|} = r^{|\pi(A)|}$  maps  $\text{col}: V(K) \rightarrow [r]$  with  $A \leq M^s(\text{col})$ . Equivalently,

$$\sum_{A \leq B} \mu(A, B) r^{|\pi(B)|} = \chi(K, r, s, A)$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where  $A = \widehat{0}$ .  $\square$

The defining rules for the Möbius function of the poset  $L^s(K)$  [12, 3.7]

- $\mu(S, S) = 1$  for all  $S \in L^s(K)$
- $\sum_{R \leq S \leq T} \mu(R, S) = 0$  when  $R \not\leq T$
- $\mu(R, S) = 0$  when  $R \not\leq S$

imply that  $\mu(\widehat{0}, \widehat{0}) = 1$  and  $\mu(\widehat{0}, \{\sigma\}) = -1$  for every  $s$ -simplex  $\sigma \in F^s(K)$ .

**Corollary 2.26.** *The highest degree terms of the  $s$ -chromatic polynomial are*

$$\chi^s(K, r) = r^{m(K)} - f_s(K) r^{m(K)-s} + \dots$$

Thus the  $s$ -chromatic polynomial determines  $f_0(K)$  and  $f_s(K)$ .

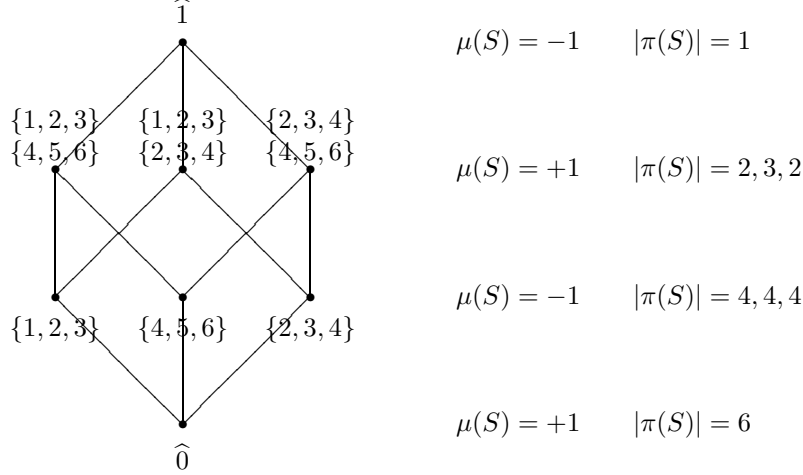


*Proof.* The  $s$ -chromatic polynomial is

$$\chi^s(K, r) = \mu(\widehat{0}, \widehat{0})r^{f_0(K)} + \sum_{\sigma \in F^s(K)} \mu(\widehat{0}, \{\sigma\})r^{f_0(K)-s} + \dots$$

where  $\mu(\widehat{0}, \widehat{0}) = 1$  and  $\mu(\widehat{0}, \{\sigma\}) = -1$  for all  $s$ -simplices  $\sigma$  of  $K$ .  $\square$

**Example 2.27.** Consider the 2-dimensional complex  $K$  from Example 2.9. The 2-chromatic lattice  $L^2(K)$  of  $K$

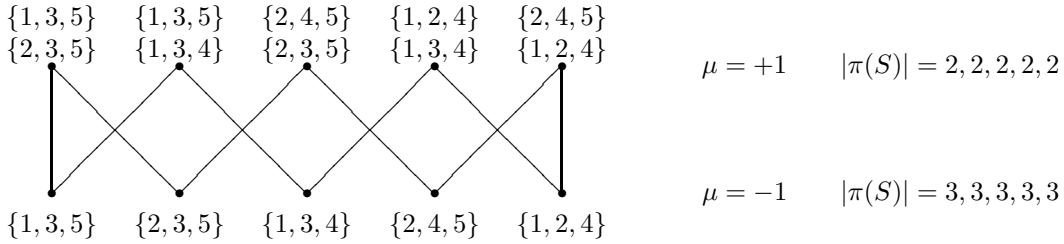


consists of *all* subsets of  $F^2(K)$ . The 2-chromatic polynomial is

$$\chi^2(K, r) = r^6 - r^4 - r^4 - r^4 + r^2 + r^3 + r^2 - r = r^6 - 3r^4 + r^3 + 2r^2 - r$$

$K$  has  $\chi^2(K, 2) = 30$  (2, 2)-colorings and  $\chi^2(K, 3) = 528$  (3, 2)-colorings.

**Example 2.28.** The triangulation MB of the Möbius band with  $f$ -vector  $f(\text{MB}) = (5, 10, 5)$  shown in Figure 1 has the following (reduced) 2-chromatic lattice  $L^2(\text{MB}) - \{\widehat{0}, \widehat{1}\}$



and 2-chromatic polynomial

$$\chi^2(\text{MB}, r) = r^5 - 5r^3 + 5r^2 - r$$

The lattice  $L^2(\text{MB})$  is graded but it is still not semi-modular [12, Proposition 3.3.2]: The meet and join of  $a = \{\{2, 3, 5\}\}$  and  $b = \{\{1, 3, 4\}\}$  are  $a \wedge b = \widehat{0}$  and  $a \vee b = \widehat{1}$ . Thus  $a$  and  $b$  cover  $a \wedge b$  but  $a \vee b$  covers neither  $a$  nor  $b$ .

**Example 2.29.** Let MT be Möbius's minimal triangulation of the torus with  $f$ -vector  $f(\text{MT}) = (7, 21, 14)$  and P2 the triangulation of the projective plane with  $f$ -vector  $f(\text{P2}) = (1, 6, 15, 10)$  shown in Figure 2 (decorated with (3, 2)-colorings). The chromatic polynomials of these two simplicial complexes are

$$\begin{aligned} \chi^1(\text{MT}, r) &= [r]_7, & \chi^2(\text{MT}, r) &= r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r \\ \chi^1(\text{P2}, r) &= [r]_6, & \chi^2(\text{P2}, r) &= r^6 - 10r^4 + 15r^3 - 6r^2 \end{aligned}$$

In both cases, the 1-skeleton is the complete graph on the vertex set. The chromatic numbers are  $\text{chr}^1(\text{MT}) = 7$ ,  $\text{chr}^1(\text{P2}) = 6$ , and  $\text{chr}^2(\text{MT}) = 3 = \text{chr}^2(\text{P2})$ .

The chromatic polynomials of simple graphs (the 1-chromatic polynomials of simplicial complexes) are known to have these properties:

- The coefficients are sign-alternating [10, §7, Corollary]
- The coefficients are log-concave (Definition 2.43) in absolute value [7]
- There are no negative roots and no roots between 0 and 1 [14]

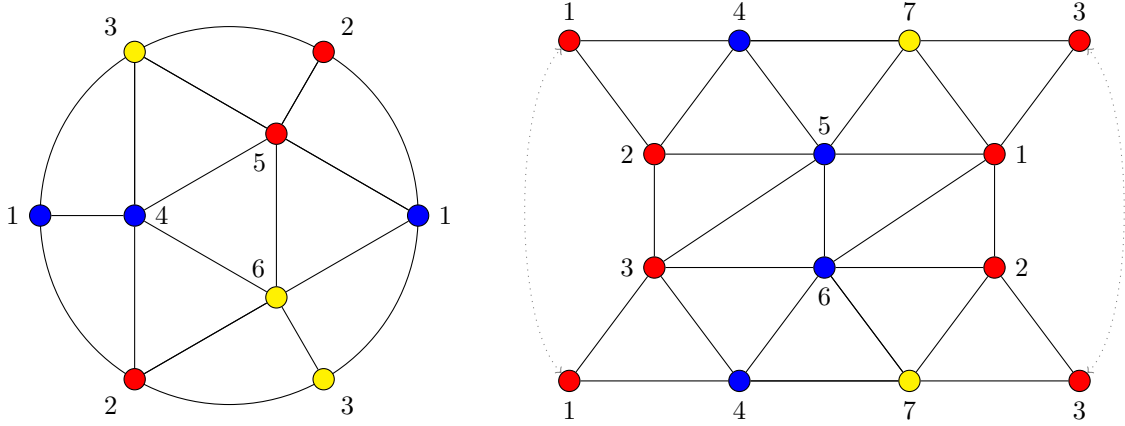


FIGURE 2. (3, 2)-colorings of P2 and MT

In contrast, the coefficients of the 2-chromatic polynomial

$$\chi^2(\text{MT}, r) = r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r = [r]_3(r+1)(r^3 + 2r^2 - 9r + 3)$$

are not sign-alternating, not log-concave in absolute value, and the polynomial has a negative root and a root between 0 and 1.

**2.4. The  $s$ -chromatic polynomial in falling factorial form.** Theorem 1.2 provides an interpretation of the coefficients of the falling factorial  $[r]_i$  in the  $s$ -chromatic polynomial of the simplicial complex  $K$ .

**Definition 2.30.**  $S(K, r, s)$  is the number of partitions of  $V(K)$  into  $r$   $s$ -independent blocks.

We think of  $S(K, r, s)$  as an  $s$ -Stirling number of the second kind for the simplicial complex  $K$ . If  $s > \dim(K)$ , then there are no  $s$ -simplices in  $K$  and all partitions of  $V(K)$  are  $s$ -independent, so that  $S(K, r, s)$  is the Stirling number of the second kind  $S(m(K), r)$  [12, p 33]. We now explain the general relation between these simplicial Stirling numbers  $S(K, r, s)$  and the usual Stirling numbers of the second kind.

Define the  $s$ -monochrome set of a partition  $P$  of  $V(K)$  to be the set

$$M^s(P) = \{\sigma \in F^s(K) \mid \sigma \text{ is contained in a block of } P\}$$

of all  $s$ -simplices entirely contained in one of the blocks of  $P$ . The set  $M^s(P)$  is an element of the  $s$ -chromatic lattice  $L^s(K)$  by Lemma 2.24.

**Theorem 2.31.** The number of partitions of  $V(K)$  into  $r$   $s$ -independent blocks is

$$S(K, r, s) = \sum_{T \in L^s(K)} \mu(\hat{0}, T) S(|\pi(T)|, r)$$

where  $\mu$  the Möbius function for the  $s$ -chromatic lattice  $L^s(K)$ .

*Proof.* For any  $B \in L^s(K)$ , let  $S(K, r, s, B)$  be the number of partitions  $P$  of  $V(K)$  into  $r$  blocks with monochrome set  $M^s(P) = B$ . We want to determine  $S(K, r, s, \emptyset) = S(K, r, s)$ . For any  $A \in L^s(K)$ ,

$$S(|\pi(A)|, r) = \sum_{A \leq B} S(K, r, s, B)$$

because there are  $S(|\pi(A)|, r)$  partitions  $P$  of  $V(K)$  into  $r$  blocks with  $A \leq M^s(P)$ . Equivalently,

$$\sum_{A \leq B} \mu(A, B) S(|\pi(B)|, r) = S(K, r, s, A)$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where  $A = \hat{0}$ .  $\square$

*Proof of Theorem 1.2.* We simply follow the proof of the similar statement for chromatic polynomials for graphs [9, Theorem 15]. When  $r \geq i$  we can get an  $(r, s)$ -coloring out of one of the  $S(K, i, s)$  partitions of  $V(K)$  into  $i$   $s$ -independent blocks by choosing  $i$  out of the  $r$  colors and assigning them to the  $i$  blocks. There are  $\binom{r}{i}$  ways of

choosing the  $i$  out of  $r$  colors and  $i!$  ways of coloring  $i$  blocks in  $i$  colors. The number of  $(r, s)$ -colorings of  $K$  in exactly  $i$  colors is thus

$$S(K, i, s) \binom{r}{i} i! = S(K, i, s) [r]_i$$

so that

$$\chi^s(K, r) = \sum_{i=1}^{m(K)} S(K, i, s) [r]_i$$

is the total number of  $(r, s)$ -colorings of  $K$ .  $\square$

**Corollary 2.32.** *The reduced Euler characteristic of the open interval  $(\widehat{0}, \widehat{1})$  in  $s$ -chromatic lattice  $L^s(K)$  is*

$$\mu(L^s(K))(\widehat{0}, \widehat{1}) = \sum_{i=\text{chr}^s(K)}^{m(K)} (-1)^{i-1} (i-1)! S(K, i, s)$$

*Proof.* Equate the terms of degree 1 of the two expressions

$$(2.33) \quad \sum_{T \in L^s(K)} \mu(\widehat{0}, T) r^{|\pi(T)|} = \sum_{i=\text{chr}^s(K)}^{m(K)} S(K, i, s) [r]_i$$

from Theorem 2.25 and Theorem 1.2 for the  $s$ -chromatic polynomial of  $K$ .  $\square$

We observe that

$$\sum_i S(K, i, s) [r]_i = \sum_i \sum_T \mu(\widehat{0}, T) S(|\pi(T)|, i) [r]_i = \sum_T \mu(\widehat{0}, T) \sum_i S(|\pi(T)|, i) [r]_i = \sum_T \mu(\widehat{0}, T) r^{|\pi(T)|}$$

so that Theorem 2.31 implies Theorem 1.2.

The  $s$ -chromatic number of  $K$  is immediately visible with the  $s$ -chromatic polynomial in factorial form because

$$\text{chr}^s(K) = \min\{i \mid S(K, i, s) \neq 0\}$$

is the lowest degree of the nonzero terms. The positive integer sequence

$$\chi^s(K, \text{chr}^s(K)), \dots, \chi^s(K, m(K)) = 1$$

has no internal zeros. (If there is a partition of  $V(K)$  into  $r$  blocks not containing any  $s$ -simplex of  $K$  and  $r < m(K)$ , then split one of the blocks with more than one vertex into two sub-blocks to get a partition of  $V(K)$  into  $r+1$  blocks containing no  $s$ -simplices of  $K$ .)

The simplicial Stirling numbers satisfy the recurrence relations

$$S(K, r, s) = \sum_{\substack{\emptyset \subsetneq U \subseteq V(K) - \{v_0\} \\ V(K) - U \text{ } s\text{-independent}}} S(K \cap D[U], r-1, s), \quad S(K, 1, s) = \begin{cases} 1 & s > \dim(K) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

To see this, fix a vertex  $v_0$  of  $K$ . Let  $P$  be partition of  $V(K)$  into  $r$   $s$ -independent subsets. Let  $U_0$  be the block containing  $v_0$ . The other blocks in  $P$  form a partition  $P_0$  of  $K \cap D[V(K) - U_0]$  into  $r-1$   $s$ -independent subsets. The map  $P \leftrightarrow (P_0, U_0)$  is a bijection.

The familiar recurrence relation  $S(m, r) = S(m-1, r-1) + rS(m-1, r)$  for Stirling numbers of the second kind does not readily apply to simplicial Stirling numbers. The closest analogue may be

$$S(K, r, s) = S(K \cap D[V(K) - \{v_0\}], r-1, s) + \sum_{P \in \mathcal{S}(K \cap D[V(K) - \{v_0\}], r, s)} |\{B \in P \mid B \cup \{v_0\} \text{ is } s\text{-independent in } K\}|$$

where  $v_0$  is a vertex of  $K$  and  $\mathcal{S}(K \cap D[V(K) - \{v_0\}], r, s)$  is the set of partitions  $P$  of the vertex set of  $K \cap D[V(K) - \{v_0\}]$  into  $r$   $s$ -independent subsets.

**Proposition 2.34.** *Let  $K$  be a subcomplex of  $L$  and assume that  $V(K) = V(L)$ .*

- (1)  $S(K, r, s) \geq S(L, r, s)$  for all  $r$ .
- (2) If  $S(K, r, s) = S(L, r, s)$  for some  $r$  with  $\frac{1}{s}(|V| - 1) \leq r \leq |V| - s$ , then  $K^s = L^s$ .

$\begin{pmatrix} 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 6 & 1 \\ 0 & 7 & 6 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 15 & 10 & 1 \\ 0 & 10 & 25 & 10 & 1 \\ 0 & 15 & 25 & 10 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 15 & 45 & 15 & 1 \\ 0 & 10 & 75 & 65 & 15 & 1 \\ 0 & 25 & 90 & 65 & 15 & 1 \\ 0 & 31 & 90 & 65 & 15 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 105 & 105 & 21 & 1 \\ 0 & 0 & 175 & 315 & 140 & 21 & 1 \\ 0 & 35 & 280 & 350 & 140 & 21 & 1 \\ 0 & 56 & 301 & 350 & 140 & 21 & 1 \\ 0 & 63 & 301 & 350 & 140 & 21 & 1 \end{pmatrix}$

TABLE 2. Chromatic tables for complete simplices  $D[m]$  for  $m = 2, \dots, 7$ 

*Proof.* (1) Let  $V$  be the vertex set of  $K$  and  $L$ . Write  $\mathcal{S}(K, r, s)$  and  $\mathcal{S}(L, r, s)$  for the set of partitions of  $V$  into  $r$  blocks containing no  $s$ -simplex of  $K$  or  $L$ , respectively. Then  $\mathcal{S}(L, r, s) \subseteq \mathcal{S}(K, r, s)$  for all  $r$  and  $s$ . Thus  $S(L, r, s) \leq S(K, r, s)$ .

(2) Suppose that  $\sigma \in F^s(L) - F^s(K)$  is an  $s$ -simplex of  $L$  that is not an  $s$ -simplex of  $K$ . Any partition of the form

$$\{\sigma\} \cup \tau, \quad \tau \in \mathcal{S}(D[V - \sigma], r - 1, s),$$

is in  $\mathcal{S}(K, r, s) - \mathcal{S}(L, r, s)$ . The set  $\mathcal{S}(D[V - \sigma], r - 1, s)$  is nonempty when

$$\text{chr}^s(D[V - \sigma]) = \left\lceil \frac{|V| - s - 1}{s} \right\rceil \leq r - 1 \leq |V| - s - 1$$

and thus  $S(K, r, s)$  is strictly greater than  $S(L, r, s)$  when  $\frac{|V|-1}{s} \leq r \leq |V| - s$ .  $\square$

**Remark 2.35** ( $S(K, r, s)$  for the complete simplex  $K = D[m]$ ). For any finite set  $M$ , let  $S(M, r, s)$  stand for  $S(D[M], r, s)$  (Definition 2.30), the number of partitions of the set  $M$  into  $r$  blocks containing at most  $s$  elements. Let us even write  $S(m, r, s)$  in case  $M = [m]$ ,  $m \geq 1$ ,  $r, s \geq 0$ . Clearly,  $S(m, r, s)$  is nonzero only when  $m/s \leq r \leq m$ . Also,  $S(m, r, s) = S(m, r)$  when  $r$  is among the  $s$  numbers  $m - s + 1, \dots, m$ . The recurrence relation

$$S(m, r, s) = \sum_{j=m-s}^{m-1} \binom{m-1}{j} S(j, r-1, s)$$

can be used to compute these numbers. Table 2 shows  $S(m, r, s)$  for small  $m$ ; the number  $S(m, r, s)$  is in row  $s$  and column  $r$  in the chromatic table (Definition 2.39) for  $D[m]$ . All the red numbers are usual Stirling numbers of the second kind.

According to Theorem 1.2, the numbers  $S(m, r, s)$  determine the  $s$ -chromatic polynomial in falling factorial form of the complete simplex on  $m$  vertices

$$\chi^s(D[m], r) = \sum_{i=\lceil m/s \rceil}^m S(m, i, s) [r]_i$$

and, according to Corollary 2.32, they also determine the reduced Euler characteristic

$$\mu_m^s(1^m)(\widehat{0}, \widehat{1}) = \sum_{i=\lceil m/s \rceil}^m (-1)^{i-1} (i-1)! S(m, i, s)$$

of the  $s$ -chromatic lattice  $L^s(D[m])$ .

More generally, if  $w: M \rightarrow \mathbf{N}$  is a function on  $M$  with natural numbers as values, let  $S(M, w, r, s)$  be the number of partitions of  $M$  into admissible blocks, where we declare a block admissible if it is a singleton or it has weight at most  $s$ . (Then  $S(m, r, s) = S([m], 1^m, r, s)$  occur when  $M = [m]$  and  $w = 1^m$  places weight 1 on all elements.) Any such partition is a partition of  $M$  into blocks of weight at most  $s$ , and therefore  $S(M, w, r, s) \leq S(\#M, r, s)$ . In particular,  $S(M, w, r, s)$  is nonzero only when  $\#M/s \leq r \leq \#M$ . The recurrence relation

$$S(M, w, r, s) = \sum_{\substack{\emptyset \neq J \subset M - \{\max(M)\} \\ M - J \text{ admissible}}} S(J, w|_J, r-1, s)$$

provides a means to compute these numbers.

The weighted version of Equation (2.33) for  $K = D[m]$ ,

$$\sum_{\sigma \in L_m^s(w)} \mu_m^s(w)(\widehat{0}, \sigma) r^{|\sigma|} = \sum_{i=\lceil m/s \rceil}^m S([m], w, i, s)[r]_i$$

implies, by equating coefficients of first degree terms, the expression

$$(2.36) \quad \mu_m^s(w)(\widehat{0}, \widehat{1}) = \sum_{i=\lceil m/s \rceil}^m (-1)^{i-1} (i-1)! S([m], w, i, s)$$

for the Euler characteristic of the weighted lattice  $L_m^s(w)$  from Remark 2.19.

Because any simplicial complex  $K$  is a subcomplex of the complete simplex  $D[m(K)]$  on its vertex set, we have

$$(2.37) \quad S(m(K), r) \geq S(K, r, s) \geq S(m(K), r, s), \quad 1 \leq r \leq m(K)$$

Moreover, these inequalities are equalities for the  $s$  highest values  $m(K) - s + 1, \dots, m(K)$  of  $r$ . Thus the  $s$  terms of highest falling factorial degree in the  $s$ -chromatic polynomial of  $K$

$$\chi^s(K, r) = \sum_{i=0}^{m(K)-s} S(K, i, s)[r]_i + \sum_{i=m(K)-s+1}^{m(K)} S(m(K), i)[r]_i$$

are given by the  $s$  Stirling numbers  $S(m(K), m(K)-s+1), \dots, S(m(K), m(K))$  of the second kind. These coefficients depend only on the size of the vertex set of  $K$ . We shall next show that the coefficient number  $s+1$  counted from above,  $S(K, m(K)-s, s)$ , informs about the number  $f_s(K)$  of  $s$ -simplices in  $K$ .

**Proposition 2.38.**  $S(K, m(K)-s, s) = S(m(K), m(K)-s) - f_s(K)$ . If  $S(K, m(K)-s, s) = S(m(K), m(K)-s, s)$  then  $K^s = D[m(K)]^s$ .

*Proof.* The only partitions of the  $S(m, m-s)$  partitions of  $V(K)$  into  $m-s$  blocks that are not  $s$ -independent are those consisting of one  $s$ -simplex of  $K$  together with singleton blocks. If  $S(K, m(K)-s, s) = S(D[m(K)], m(K)-s, s)$  then  $f_s(K) = f_s(D[m(K)])$  so  $K^s = D[m(K)]^s$ . (This is a special case of Proposition 2.34.(2).)  $\square$

**Definition 2.39.** The chromatic table,  $\chi(K)$ , of  $K$  is the  $(\dim(K) \times m(K))$ -table with  $S(K, r, s)$  in row  $s$  and column  $r$ .

This means that row  $s$  in the chromatic table lists the coefficients of the  $s$ -chromatic polynomial. The chromatic table of a 3-dimensional simplicial complex  $K$ , for instance, looks like this

	$r=1$	$r=2$	$\dots$	$r=m-3$	$r=m-2$	$r=m-1$	$r=m$
$S(K, \cdot, 1)$	$S(K, 1, 1)$	$S(K, 2, 1)$	$\dots$	$S(K, m-3, 1)$	$S(K, m-2, 1)$	$S(m, m-1) - f_1$	$S(m, m) = 1$
$S(K, \cdot, 2)$	$S(K, 1, 2)$	$S(K, 2, 2)$	$\dots$	$S(K, m-3, 2)$	$S(m, m-2) - f_2$	$S(m, m-1)$	$S(m, m) = 1$
$S(K, \cdot, 3)$	$S(K, 1, 3)$	$S(K, 2, 3)$	$\dots$	$S(m, m-3) - f_3$	$S(m, m-2)$	$S(m, m-1)$	$S(m, m) = 1$

where the red entries in row  $s$  are Stirling numbers of the second kind  $S(m, r)$  for  $m-s+1 \leq r \leq m$ , and the blue entry in row  $s$  is  $S(m(K), m(K)-s) - f_s(K)$ .

**Example 2.40.** The chromatic tables of the 2-dimensional simplicial complexes from Examples 2.9, 2.28, and 2.29 are

$$\begin{aligned} \chi(K) &= \begin{pmatrix} 0 & 0 & 2 & 10 & \textcolor{blue}{7} & \textcolor{red}{1} \\ 0 & 15 & 73 & \textcolor{blue}{62} & \textcolor{red}{15} & \textcolor{red}{1} \end{pmatrix} & \chi(\text{MB}) &= \begin{pmatrix} 0 & 0 & 0 & \textcolor{blue}{0} & \textcolor{red}{1} \\ 0 & 5 & \textcolor{blue}{20} & \textcolor{red}{10} & \textcolor{red}{1} \end{pmatrix} \\ \chi(\text{MT}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \textcolor{blue}{0} & \textcolor{red}{1} \\ 0 & 0 & 84 & 231 & \textcolor{blue}{126} & \textcolor{red}{21} & \textcolor{red}{1} \end{pmatrix} & \chi(\text{P2}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & \textcolor{blue}{0} & \textcolor{red}{1} \\ 0 & 0 & 45 & \textcolor{blue}{55} & \textcolor{red}{15} & \textcolor{red}{1} \end{pmatrix} \end{aligned}$$

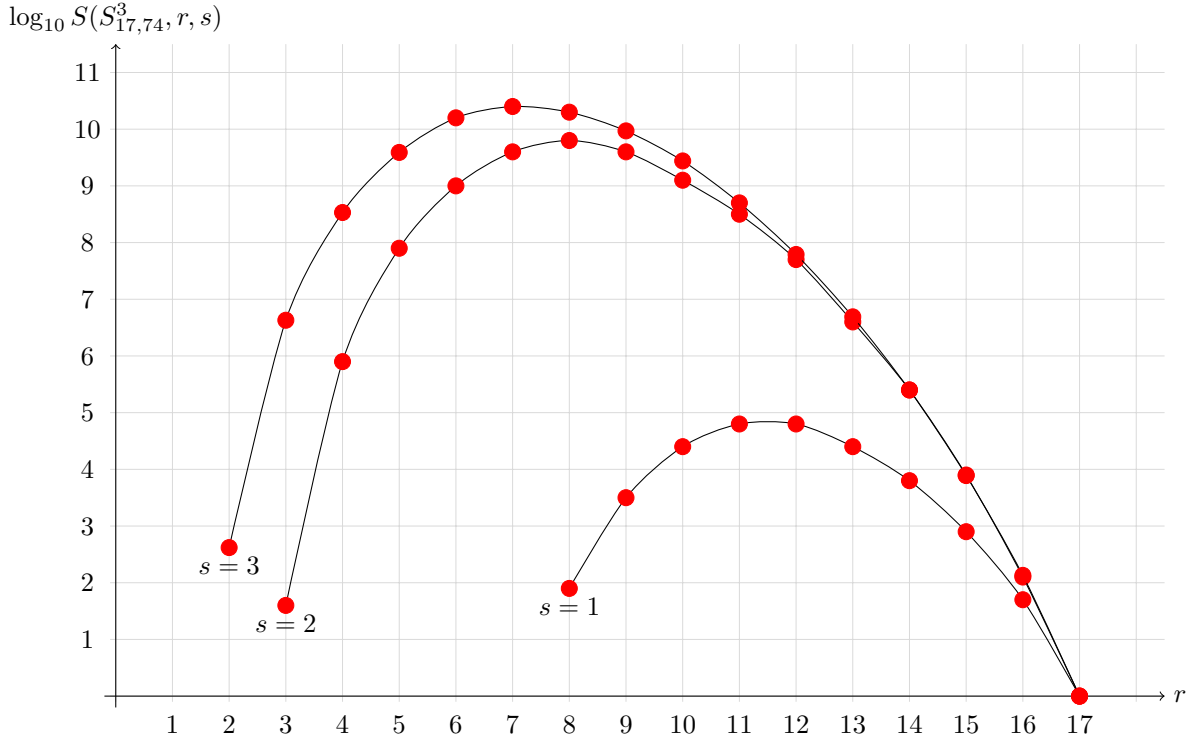
The red entries in column  $r$  are Stirling numbers  $S(m, r)$  and they are independent of the row index. The blue entry in row  $s$  and column  $m-s$ , which equals  $S(m-s, s) - f_s(K)$ , detects if  $K$  has maximal  $s$ -skeleton by Proposition 3.

**Example 2.41.** Let  $K = \text{AS3}$  be Altshuler's peculiar triangulation of the 3-sphere with  $f$ -vector  $f = (10, 45, 70, 35)$  [1]. The 1-chromatic polynomial is  $\chi^1(\text{AS3}, r) = [r]_{10}$  as  $K^1$  is the complete graph on 10 vertices. The chromatic table is

$$\chi(\text{AS3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \textcolor{blue}{0} & \textcolor{red}{1} \\ 0 & 0 & 0 & 1360 & 8475 & 10355 & 4200 & \textcolor{blue}{680} & \textcolor{red}{45} & \textcolor{red}{1} \\ 0 & 26 & 4320 & 25915 & 38550 & 22152 & \textcolor{blue}{5845} & \textcolor{red}{750} & \textcolor{red}{45} & \textcolor{red}{1} \end{pmatrix}$$

The blue numbers determine the  $f$ -vector

$$f(\text{AS3}) = (10, S(10, 9) - \chi(\text{AS3})_{19}, S(10, 8) - \chi(\text{AS3})_{28}, S(10, 7) - \chi(\text{AS3})_{37})$$

FIGURE 3. The simplicial Stirling numbers for  $S_{17,74}^3$ 

The row numbers of the first nonzero term in each row tell us that  $\text{chr}^1(\text{AS3}) = 10$ ,  $\text{chr}^2(\text{AS3}) = 4$ , and  $\text{chr}^3(\text{AS3}) = 2$ .

**Example 2.42.** The nonconstructible, nonshellable 3-sphere  $S_{17,74}^3$ ,  $f = (17, 91, 148, 74)$ , found by Lutz [8], has

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$
$s = 1$	0	0	0	0	0	0	0	88	3089
$s = 2$	0	0	36	702475	82949364	1075420155	3827766587	5493687086	3876597169
$s = 3$	0	422	4319865	338438489	3903094622	14292381565	22946854806	19158310796	9202775199

	$r = 10$	$r = 11$	$r = 12$	$r = 13$	$r = 14$	$r = 15$	$r = 16$	$r = 17$
$s = 1$	23017	55285	54973	25941	6210	762	45	1
$s = 2$	1507939074	346346664	48855523	4302470	235026	7672	136	1
$s = 3$	2708454744	507528561	61784524	4903589	249826	7820	136	1

as its chromatic table. Figure 3 shows a semi-logarithmic plot of the simplicial Stirling numbers  $S(S_{17,74}^3, r, s)$ . The triangulation  $\Sigma_{16}^3$ ,  $f = (16, 106, 180, 90)$ , of the Poincaré homology 3-sphere constructed by Björner and Lutz [2, Theorem 5] has

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$
$s = 1$	0	0	0	0	0	0	0	0
$s = 2$	0	0	0	4589	2974411	69671411	300475213	442354547
$s = 3$	0	3	845561	70005500	701299653	2158716508	2888730959	2000811501

	$r = 9$	$r = 10$	$r = 11$	$r = 12$	$r = 13$	$r = 14$	$r = 15$	$r = 16$
$s = 1$	0	0	0	0	28	44	14	1
$s = 2$	292864435	100793551	19546606	2225261	150095	5840	120	1
$s = 3$	792553648	190527025	28730056	2750278	165530	6020	120	1

as its chromatic table.

Observe that all the above chromatic tables have strictly log-concave rows.

**Definition 2.43.** [11] A finite sequence  $a_1, a_2, \dots, a_N$  of  $N \geq 3$  nonnegative integers is strictly log-concave if  $a_{i-1}a_{i+1} < a_i^2$  for  $1 < i < N$  (and log-concave if  $a_{i-1}a_{i+1} \leq a_i^2$ ).

It has been conjectured that the sequence of coefficients of the 1-chromatic polynomial of a simple graph in falling factorial form,  $r \rightarrow S(K, 1, r)$ ,  $\text{chr}^1(K) \leq r \leq m(K)$ , is log-concave [4, Conjecture 3.11]. More generally, one may ask

**Question 2.44.** *Is the finite sequence of simplicial Stirling numbers*

$$r \rightarrow S(K, r, s), \quad \text{chr}^s(K) \leq r \leq m(K),$$

log-concave for fixed  $K$  and  $s$ ?

This seems to be the right question to ask as it may be true for *all* the chromatic polynomials of a simplicial complex and we have seen that the absolute value of the coefficients of the  $s$ -chromatic polynomial are simply not log-concave for  $s > 1$ .

Note that the Stirling numbers of the second kind, which are upper bounds for the simplicial Stirling numbers  $S(K, r, s)$  by the inequalities (2.37), are log-concave in  $r$  [11, Corollary 2].

We shall now examine Question 2.44 on two spherical boundary complexes of cyclic  $n$ -polytopes.

**Definition 2.45.**  $\partial\text{CP}(m, n)$ ,  $m > n$ , is the  $(n-1)$ -dimensional simplicial complex on the ordered set  $[m]$  with the following facets: An  $n$ -subset  $\sigma$  of  $[m]$  is a facet if and only if between any two elements of  $[m] - \sigma$  there is an even number of vertices in  $\sigma$ .

By Gale's Evenness Theorem [6], the simplicial complex  $\partial\text{CP}(m, n)$  triangulates the boundary of the cyclic  $n$ -polytope on  $m$  vertices. Thus  $\partial\text{CP}(m, n)$  is a simplicial  $(n-1)$ -sphere on  $m$  vertices and it is  $\lfloor n/2 \rfloor$ -neighborly in the sense that  $\partial\text{CP}(m, n)$  has the same  $s$ -skeleton as the full simplex on its vertex set when  $s < \lfloor n/2 \rfloor$ .

**Example 2.46** (Cyclic polytopes with log-concave simplicial Stirling numbers of the second kind). Let  $\partial\text{CP}(m, n)$  be the triangulated boundary of the cyclic polytope on  $m$  vertices in  $\mathbf{R}^n$ . The simplicial complex  $\partial\text{CP}(m, n)$  is an  $m$ -vertex triangulation of  $S^{n-1}$ . The chromatic tables of the simplicial 3-spheres  $\partial\text{CP}(m, 4)$  on  $m = 6, 7, 8, 9, 10$  vertices are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 21 & 47 & 15 & 1 \\ 0 & 16 & 81 & 65 & 15 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 28 & 147 & 112 & 21 & 1 \\ 0 & 21 & 238 & 336 & 140 & 21 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 50 & 393 & 582 & 226 & 28 & 1 \\ 0 & 29 & 654 & 1533 & 1030 & 266 & 28 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 94 & 1062 & 2523 & 1719 & 408 & 36 & 1 \\ 0 & 36 & 1729 & 6471 & 6591 & 2619 & 462 & 36 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 180 & 2980 & 10200 & 10777 & 4225 & 680 & 45 & 1 \\ 0 & 46 & 4445 & 25960 & 38550 & 22152 & 5845 & 750 & 45 & 1 \end{pmatrix}$$

All rows are strictly log-concave. As  $\partial\text{CP}(m, 4)^1 = D[m]^1$ , the 1-chromatic number  $\text{chr}^1(\partial\text{CP}(m, 4)) = m$ , and it is not difficult to see that the 2-chromatic number  $\text{chr}^2(\partial\text{CP}(m, 4))$  is 2 if  $m$  is even and 3 if  $m$  is odd [5].

Right multiplication with the upper triangular matrix  $([j]_i)_{1 \leq i, j \leq m(K)}$  with  $[j]_i = \binom{j}{i} i! = \frac{j!}{(i-j)!}$  in row  $i$  and column  $j$  transforms, by Theorem 1.2, the chromatic table into the  $(\dim(K) \times m(K))$ -matrix

$$\chi(K)([j]_i)_{1 \leq i, j \leq m(K)} = (\chi^s(K, i))_{\substack{1 \leq s \leq \dim(K) \\ 1 \leq i \leq m(K)}}$$

with the  $m(K)$  values  $\chi^s(K, i)$ ,  $1 \leq i \leq m(K)$ , of the  $s$ -chromatic polynomial in row  $s$ . This matrix of chromatic polynomial values appears also to have log-concave rows.

### 3. CHROMATIC UNIQUENESS

In this section we briefly discuss to what extent simplicial complexes are determined by their chromatic polynomials. Proposition shows that the chromatic table of a simplicial complex determines its  $f$ -vector.

**Definition 3.1.**  $K$  is chromatically unique if it is determined up to isomorphism by its chromatic table.

In Lemma 3.2 below,  $K \amalg L$  is the disjoint union and  $K \vee L$  the one-point union of  $K$  and  $L$ . The proof is identical to the one for the similar statements about chromatic polynomials for simple graphs.

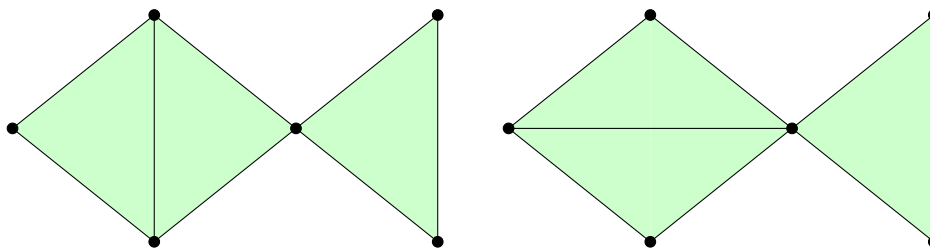
**Lemma 3.2.** If  $K$  and  $L$  are finite simplicial complexes then

$$\chi^s(K \amalg L, r) = \chi^s(K, r) \chi^s(L, r), \quad \chi^s(K \vee L, r) = \frac{\chi^s(K, r) \chi^s(L, r)}{r}$$

for all  $r$  and all  $s \geq 0$ .

The two nonisomorphic simplicial complexes





are not chromatically unique as they have identical chromatic tables

$$\begin{pmatrix} 0 & 0 & 2 & 10 & 7 & 1 \\ 0 & 15 & 73 & 62 & 15 & 1 \end{pmatrix}$$

by Lemma 3.2. (These two complexes are, however, PL-isomorphic.)

On the other hand, Proposition 2.34.(2) immediately implies that the  $s$ -skeleton of a full simplex is chromatically unique (in a very strong sense).

**Proposition 3.3.** *If  $K$  has the same  $s$ -chromatic polynomial as a full simplex  $D[N]$ , then  $K$  and  $D[N]$  have isomorphic  $s$ -skeletons.*

*Proof.* If  $K$  and  $D[N]$  have the same  $s$ -chromatic polynomial for some  $s \geq 1$ , then  $K$  has  $N$  vertices (Corollary 2.26), and, since  $\chi^s(K, N-s) = \chi^s(D[N], N-s)$ , the  $s$ -skeleton of  $K$  is isomorphic to the  $s$ -skeleton of the full simplex on  $N$  vertices (Proposition 2.34.(2)).  $\square$

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